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3D Stress Analysis of Functionally Graded Plate under Static Loading Condition

Mihir P. Mahajan[1] and Sandeep S. Pendhari[2]

[1] *Structural Engineering Department, Veermata Jijabai Technological Institute, Mumbai 400019, India.*
mihirmahajan23@gmail.com

[2] *Structural Engineering Department, Veermata Jijabai Technological Institute, Mumbai 400019, India.*
sspendhari@vjti.org.in

Abstract- Three dimensional (3D) stress analysis is performed in this paper for functionally graded (FG) plate using semi analytical approach developed by Kant *et al.* [1]. Modulus of elasticity is assumed to be varied according to power law through the thickness of plate. The mathematical model consists in defining a two-point boundary value problem (BVP) governed by a set of coupled first-order ordinary differential equations (ODEs) in the plate thickness direction. Analytical solutions available in the literature is used to show the accuracy, simplicity and effectiveness of present semi analytical solution.

Keywords- Functionally graded plates, Semi-analytical method, Power law, Boundary value problem.

I. INTRODUCTION

Functionally Graded Materials (FGM) are a class of advanced material composites that have continuous and smooth variation of material properties from one surface to another .One of the advantages of a monotonous variation of volume fraction of constituent phases is the elimination of stress discontinuity that is often encountered in laminated composites and accordingly, reducing thermal stresses, residual stresses, stress concentrations and avoiding delamination-related problems. These materials are marked by a heterogeneous composition and are typically made of isotropic components such as metals and ceramics.

Comprehensive work on bending analysis of functionally graded plate subjected to mechanical loading are presented by many researchers in literature. Saidi *et al.* [2] has presented an exact analytical solution for static analysis of FG annular sector plates by assuming that the radial edges of plate are simply supported and arbitrary boundary conditions are assumed along the circular edges. Hosseini-Hashemi *et al.* [3] presented exact closed form solution for free vibration analysis of Levy's type rectangular FG plate with different boundary conditions using first-order shear deformation theory (FOSDT).

A hybrid quasi-3D hyperbolic shear deformation theory for static and free vibration analysis of functionally graded plates is presented by Neves *et al.* [4]. Belabed *et al.* [5] derived analytical solution for bending and free vibration analysis for simply supported FGM plates based on new efficient and simple higher order shear and normal deformation theory which accounts for the stretching and shear deformation effects without requiring a shear correction factor. Abdelaziz *et al.* [6] studied the two variable refined theory (RPT) for isotropic plates and (2006) for orthotropic plates. They extended the RPT for static response of FGM sandwich plates and also obtained closed form solutions for simply supported FGM sandwich plates using Navier solution.

An analysis based on shear deformation theories for FGM plate are presented by several authors, Wu and Li [7] presented solution using a RMVT-based third order shear deformation theory (TSDT). Wu *et al.* [8] used RMVT-based meshless collocation and element-free Galerkin methods for the quasi-3D analysis of multilayered composite and FGM plates. Zenkour [9] used a generalized HSDT (Touratier's HSDT). Mantari *et al.* [10] used HSDT.

An effort is put in this paper to reformulate the semi analytical model developed by Kant *et al.* [1] for stress analysis of simply (diaphragm) supported FG pate under transverse loading. The formulation is based on defining two-point boundary value problem (BVP) along the thickness directions.

II. MATHEMATICAL FORMULATION

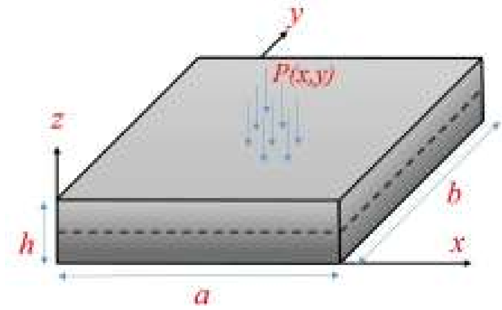

Figure 1. Functionally graded plate subjected to transverse load

A FG plate (***Figure 1***), simply supported on all its four edges is considered. A right-handed orthogonal co-ordinate system (x, y, and z) is chosen such that the plate occupies a domain Ω in the x-y plane and z-axis is normal to the plane. The top surface of a plate is loaded only with transversely distributed load and it can be expressed as,

$$p(x,y)=\sum_{mn}^{\infty} p_{0mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where, m and n =1,3,5,…….and other surfaces are free from any stresses.

The 3D equations of equilibrium are,

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+B_x=0 \qquad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+B_y=0 \qquad \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+B_z=0 \quad (1)$$

where, B_x, B_y and B_z are the body forces per unit volume in x, y and z directions, respectively

It is assumed that the Poisson's ratio is constant through the thickness and variation of Young's modulus through the plate thickness is given by

$$E(z)=E_M+(E_C-E_M)\left(\frac{2z+h}{2h}\right)^k$$

Further, it is assumed here that FG material is isotropic at every point. Therefore, constitutive relations for FG plate can be written as,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & symmetric & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix} \quad (2)$$

where, $C_{11} = C_{22} = C_{33} = \dfrac{E(z)(1-\upsilon^2)}{(1-3\upsilon^2-2\upsilon^3)}$ $\quad C_{12} = C_{13} = C_{23} = \dfrac{E(z)(\upsilon+\upsilon^2)}{(1-3\upsilon^2-2\upsilon^3)}$ $\quad C_{44} = C_{55} = C_{66} = \dfrac{E(z)}{2(1+\upsilon)}$

and E_M = Young's modulus at the bottom of the beam

E_C = Young's modulus at the top of the beam

υ = Poisson's ratio

z = Distance at which the value of the material property is to be determined

k = Parameter that dictates material variation profile through the thickness

and, general 3D linear strain-displacement relations are,

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{3}$$

The Equations (1)-(3) have a total of fifteen unknowns in fifteen equations. After a simple algebraic manipulation of the above sets of equations, a set of PDEs involving only six primary $u, v, w, \tau_{xz}, \tau_{yz}$ and σ_z dependent variables, are obtained as follows

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} + \left(\frac{C_{66}}{C_{55}C_{66}}\right)\tau_{xz} \qquad \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y} + \left(\frac{C_{55}}{C_{55}C_{66}}\right)\tau_{yz} \qquad \frac{\partial w}{\partial z} = \frac{\sigma_z}{C_{33}} - \frac{1}{C_{33}}\left(C_{31}\frac{\partial u}{\partial x} + C_{32}\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial \tau_{xz}}{\partial z} = -\left(C_{11} - \frac{C_{13}C_{31}}{C_{33}}\right)\frac{\partial^2 u}{\partial x^2} - C_{44}\frac{\partial^2 u}{\partial y^2} - \left(C_{12} + C_{44} - \frac{C_{13}C_{32}}{C_{33}}\right)\frac{\partial^2 v}{\partial x \partial y} - \frac{C_{13}}{C_{33}}\frac{\partial \sigma_z}{\partial x} - B_x$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\left(C_{21} + C_{44} - \frac{C_{23}C_{31}}{C_{33}}\right)\frac{\partial^2 u}{\partial x \partial y} - C_{44}\frac{\partial^2 u}{\partial x^2} - \left(C_{22} - \frac{C_{23}C_{32}}{C_{33}}\right)\frac{\partial^2 v}{\partial y^2} - \frac{C_{23}}{C_{33}}\frac{\partial \sigma_z}{\partial y} - B_y \qquad \frac{\partial \sigma_z}{\partial z} = -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - B_z \tag{4}$$

The above PDEs defined by Equation (4) can be further reduced to a coupled first-order ODEs by using double Fourier trigonometry series expansion for primary displacement variables satisfying the simply support end conditions on all four edges.

$$u(x,y,z) = \sum_{mn} u_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \tau_{xz}(x,y,z) = \sum_{mn} \tau_{xzmn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$v(x,y,z) = \sum_{mn} v_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad \tau_{yz}(x,y,z) = \sum_{mn} \tau_{yzmn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w(x,y,z) = \sum_{mn} w_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \sigma_z(x,y,z) = \sum_{mn} \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5}$$

Substituting Equations (5) and its derivatives into Equation (4) and noting orthogonality conditions of trigonometric functions, the following ODEs are obtained,

$$\frac{du_{mn}(z)}{dz} = -\frac{m\pi}{a} w_{mn}(z) + \frac{1}{C_{55}}\tau_{xzmn}(z) \qquad \frac{d\sigma_{zm}(z)}{dz} = \frac{m\pi}{a}\tau_{xzm}(z) + \frac{n\pi}{b}\tau_{yzm}(z) - B_z(x,y,z)$$

$$\frac{dw_{mn}(z)}{dz} = \frac{C_{31}}{C_{33}}\frac{m\pi}{a}u_{mn}(z) + \frac{C_{32}}{C_{33}}\frac{n\pi}{b}v_{mn}(z) + \frac{1}{C_{33}}\sigma_{zm}(z) \qquad \frac{dv_{mn}(z)}{dz} = -\frac{n\pi}{b}w_{mn}(z) + \frac{1}{C_{66}}\tau_{yzmn}(z)$$

$$\frac{d\tau_{xzmn}(z)}{dz} = \left(C_{11} - \frac{C_{13}C_{31}}{C_{33}}\right)\frac{m^2\pi^2}{a^2}u_{mn}(z) + C_{44}\frac{n^2\pi^2}{b^2}u_{mn}(z)$$
$$+\left(C_{12} + C_{44} - \frac{C_{13}C_{32}}{C_{33}}\right)\frac{mn\pi^2}{ab}v_{mn}(z) - \frac{C_{13}}{C_{33}}\frac{m\pi}{a}\sigma_{zm}(z) - B_x(x,y,z)$$

$$\frac{d\tau_{yzmn}(z)}{dz} = \left(C_{21} + C_{44} - \frac{C_{23}C_{31}}{C_{33}}\right)\frac{mn\pi^2}{ab}u_{mn}(z) + C_{44}\frac{m^2\pi^2}{a^2}v_{mn}(z)$$
$$+\left(C_{22} - \frac{C_{23}C_{32}}{C_{33}}\right)\frac{n^2\pi^2}{b^2}v_{mn}(z) - \frac{C_{23}}{C_{33}}\frac{n\pi}{b}\sigma_{zm}(z) - B_y(x,y,z) \qquad (6)$$

Equation (6) defines the governing equation of a two-point BVP in ODEs in the domain. $0 < z < h$ The basic approach to the numerical integration of the BVP defined in Equation (6) is to transform the given BVP into a set of initial value problems (IVPs). Numbers of successful and well-tested numerical algorithms are available in literature for solution of IVPs expressed by ODEs and are used in present study.

III. NUMERICAL STUDY

Numerical investigations on simply supported (diaphragm supported) functionally graded plate is performed to establish the accuracy of the formulation presented in the preceding sections of the paper. The 3D shear deformation theories solution available in the literature is considered as benchmark solution for comparison.

Following normalizations are used here for the uniform comparison of the results.

$$\bar{w}(z) = \frac{10E_C h^3}{q_0 a^4} w\left(\frac{a}{2},\frac{b}{2},z\right) \qquad \bar{u}(z) = \frac{100h^3 E_C}{a^4 q_0} u\left(0,\frac{b}{2},z\right)$$

$$\overline{\sigma_{xx}}(z) = \frac{h}{aq_0}\sigma_x\left(\frac{a}{2},\frac{b}{2},z\right) \qquad \overline{\sigma_{xy}}(z) = \frac{h}{aq_0}\sigma_{xy}(0,0,z) \qquad \overline{\sigma_{xz}}(z) = \frac{h}{aq_0}\tau_{xz}\left(0,\frac{b}{2},z\right)$$

A square plate of dimension 10 m with aspect ratio (*a/h*) 10 is considered here. Modulus of elasticity at plate top surface is 70 GPa (aluminum) and at bottom of plate is 380 GPa (alumina). Possion's ratio is 0.3. Plate is subjected to bi-directional sinusoidal transverse loading on the top surface. Modulus of elastcity is varied from bottom to top with power-law index parameter is 2.0. The comaprasion of present solution have been presented in ***Table 1***. Through thickness varaition of inplane displacement, inplane normal stress and transverse shear stress have been depicted for aspect ratio (*a/h*) 5 and power-law index 4.0 in ***Figure 2*** and compared with available solution. It is observed from the comparasion that present solution are in well agreement with published solution.

Table 1 Normalized inplane normal stresse, inplane and transverse shear stresses and transverse displacement of FG plate under bisinusoidal transverse load

k	Theories	$\overline{u}(-h/4)$	$\overline{w}(0)$	$\overline{\sigma_{xx}}(h/3)$	$\overline{\sigma_{xy}}(-h/3)$	$\overline{\sigma_{xz}}(h/6)$
2	HSDT [10]	0.8957	0.7564	1.3940	0.5438	0.2741
	SEMI-ANALYTICAL APPROACH [PRESENT]	0.9013	0.7570	1.3788	0.5115	0.2497
	Generalised TSDT [9]	0.9281	0.7573	1.3954	0.5441	0.2763
	RMVT-based TSDT [7]	0.8984	0.7573	1.396	0.5442	0.2491
	RMVT-based collocation [8]	0.9015	0.7572	1.4129	0.5437	0.2495
	RMVT-based Galerkin [8]	0.9013	0.7571	1.4133	0.5436	0.2495

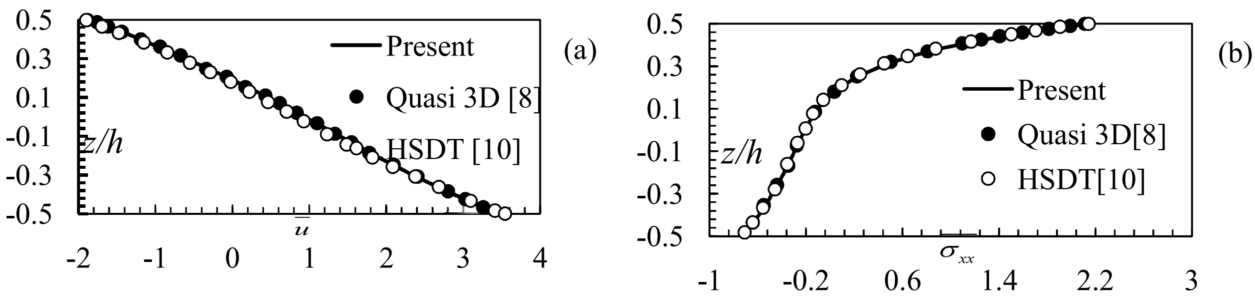

Figure2- (a)Through thickness variation of a inplane displacement $\overline{u}$,(b) inplane normal stress $\overline{\sigma_{xx}}$ for simply supported FG plate under sinusoidal load

IV. CONCLUDING REMARK

A simple semi analytical formulation presented in this paper for 3D stress analysis of FG plate with material properties varying according to power law in thickness direction. A two-point BVP governed by a set of coupled first-order ODEs is formed by assuming a chosen set of primary variables along the thickness direction of the plate which satisfy the simply (diaphragm) supported end conditions exactly. No simplifying assumptions through the thickness of the plate are introduced. Analytical solution based on shear deformation theories is used for comparison and to show the effectiveness and simplicity of the semi analytical formulation.

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