Scientific Journal Impact Factor (SJIF): 3.518
International Journal of Modern Trends in Engineering and Research
IJMTER
ISSN (ONLINE) : 2349 - 9745 ISSN (PRINT) : 2393 - 8161

INTEGRAL LIMIT THEOREMS FOR THE FIRST PASSAGE TIME OF THE MARKOV CHAIN FOR LEVEL AND THEIR APPLICATIONS

SoltanAli Aliyev[1] and Yasin Ismail Rustamov[2]
[1]*National Academy of Sciences of Azerbaijan,Institute of Mathematics and Mechanics, Baku, Azerbaijan*
[2]*National Academy of Sciences of Azerbaijan, Institute of Control Systems, Baku, Azerbaijan*

Abstract -In the present paper, under wider assumptions the integral limit theorems for the first passage time of the Markov chain for level are proved. As example of applications of obtained results one model from the theory of branching processes is considered.
Keywords- Passage time, Markov chain, random walk,limit theorems,transient probability,and branching process.

I. INTRODUCTION

Let $X=\{X_n, n\in Z^+\}$be a Markov chain with values on the real line $R=(-\infty,\infty)$ and with transient probability
$P\{X_{n+1}\in B / X_n=\upsilon\}=P_n(\upsilon,B)$,
Where $\upsilon\in R$ and $B\in\beta(R)$ is σ-algebra of Borel sets in R.

Here, homogeneity of transient probability $P_n(\upsilon,B)$ in time is not assumed. Consider the linear first passage time

$$\tau_c=\inf\{n: X_n \succ c\} \qquad (1)$$

of the Markov chain X for the level $c\ge 0$. We'll assume that $\inf\{\varnothing\}=\infty$.

Study of asymptotic properties of distributions of the first passage time is the important part of the theory of boundary value problems for random walks. The first passage time of the random process (walk) for a boundary is often used in solving the problems from applied fields of probability theory [2,8,14,16]. As it is noted in [8], the problems on estimation of reliability, finding of employment periods and waiting time, moment of loss of requirements for queneing systems, moments of devastation and overflow of store-housee in stocks control problems, duration of regeneration periods for regenerating processes and etc. are formulated in terms of the first passage time of the Markov chain for a boundary.

Development of theory of limit theorems for sequences of dependent random variables (for example, for martingales and Markov chains) and also different applied problems indicate that it is necessary to study boundary value problems for random walks in a more general statement [8,10].

Various estimations for the function of distribution of the first passage time of homogeneous Markov process with discrete time from some subset of state space are obtained in [8].

Obtaining of various estimations and also limit theorems for the distribution τ_c is stipulated by the fact that finding of exact form of the distribution τ_c is practically unsolvable problem.

Recently, there appeared some papers [3,4,15 wherein asymptotic properties of distribution of boundary functionals related with the .first passage time of the Markov chain for some boundary are studied.

In the present paper, under wider assumptions we prove integral limit theorems for the first passage time of the Markov chain of form (1). Under this one can understand any statement that under some conditions there exist the constants A(c) > 0 and B (c) such that

$$\lim_{c \to \infty} P(\tau_c - B(c) \le xA(c) = G(x)$$

at each point $x \in R$ of continuity of non-degenerate distribution function $G(x)$.

As a corollary, the integral limit theorem is obtained for the first passage time of growin level by a branching process with diffusion that enables to estimate distribution of the first propagation of epidemic process [13] (seealso [1,7]).

II. FORMULATION OF THE BASIC RESULTS

We'll adhere to the terminology of the papers [3,9,16]. Following [3] and [9], by $\xi_n(\upsilon)$ we denote a random variable whose distribution coincides with distribution of the jump of the chain X fromthe state υ at time n, i.e.

$P\{\upsilon + \xi_n(\upsilon) \in B\} = P_n(\upsilon, B)$, $\quad B \in \beta(B)$.

Notice that the family of jumps $\xi_n(\upsilon)$, $n \ge 0$ forms a sequence of random processesdetermined on one probability space[6].Introduce the followingdefinition [9,16].

Definition1.A Markov chain $X = \{X_n, n \in Z^+\}$ is said to be a chain mith asymptotic homogeneous (in time and in space) mean drift if $E\xi_n(\upsilon) \to \mu$ as $n, \upsilon \to \infty$ for some number $\mu \in R$ (therewith, existence of $E\xi_n(\upsilon)$ for all values of n and υ is not assumed).

Definition2. The sequence of random variables $X_n, n \ge 1$is called uniformly continuous in probabilityif for any $\varepsilon \succ 0$ there exists $\delta \succ 0$such that for all $n \ge 1$

$$P\left\{\max_{0 \le k \le n\delta} |X_{n+k} - X_n| \ge \varepsilon\right\} \prec \varepsilon \quad (2)$$

Noticethatanysequenceofrandomvariablesconvergingalmostsurelytofinitelimitisunifo rmlycontinuousinprobability.Wealsonoticethatifrelation(2)isfulfilledforsufficientlylarge n andfixed $\delta > 0$,thenitisfulfilledalsoforall $n \ge 1$andsmall δ [16].

Theorem1. GettheMarkovchain $X = \{X_n, n \in Z^+\}$haveanasymptoticallyhomogeneousintime andinspacemeandrift $\mu > 0$,moreover $X_n \to \infty$ almostsurelyas $n \to \infty$.Assumethatforsometime N andspatiallevel U the family of squares of jumps $\{\xi_n^2(\upsilon), n \ge N, \upsilon \ge U\}$ is integrable uniformly with respectto n and v sothatas $n, \upsilon \to \infty$

$$E\xi_n(\upsilon) = \mu + 0(1/\sqrt{n} + 1/\sqrt{\upsilon}),$$

$$D\xi_n(\upsilon) \to \sigma^2 > 0$$

And the sequence $\dfrac{X_n - n\mu}{\sigma\sqrt{n}}$, $n \ge 1$ is uniformly continuous in probability.

Then

$$\lim_{c \to \infty} P(\tau_c^* \le x) = \Phi(x), \ x \in R,$$

Where $\tau_c^* = \dfrac{\tau_c - c/\mu}{\frac{\sigma}{\mu}\sqrt{c/\mu}}$ and $\Phi(x)$ is a function of a standard normal distribution.

Theorem 2. Let theMarkov chain X_n be homogeneous in time and in space, i.e. distribution of the jump of the chain $\xi_n(\upsilon)$ doesn't depend on $n$and$v$. Assume that $X_n \uparrow \infty$ almost surely as $n \to \infty$ and distribution of the jump belongs to the attraction domain of stable distribution $G_\alpha(x)$ with characteristically exponent $\alpha \in (0,1)$. Furthermore, let for the sequence of normalizing constants $A(n) = n^{1/\alpha} L_\alpha(n), 0 < \alpha < 1$

$$\lim_{n\to\infty} P\left(\frac{X_n}{A(n)} \le x\right) = G\alpha(x),\ x > 0,$$

Where $L_\alpha(x),\ x > 0$ is a slowly changing function at infinity?

Then, if $\frac{c}{A(n)} \to x > 0$ as $n = n(c) \to \infty$ and $c \to \infty$ then

$$\lim_{n\to\infty} P\left(\frac{\tau_c}{c^\alpha L^*_{1/\alpha}(c)} \le x^{-\alpha}\right) = G_\alpha(x),$$

Where $L^*_{1/\alpha}(c)$ is function $\frac{1}{\alpha}$ - conjugated to $L_\alpha(x)$.

Notice that a model from the theory of branching processes wherein Theorem 2 is applied, is considered in section 5.

Remark 1. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables belonging to attraction domain of stable distribution with characteristically exponent $\alpha \in (1,2)$. Then, the simplest examples of the Markov chains satisfying the conditions of Theorems 1 and 2 give ordinary summation process $X_n = \sum_{i=1}^{n} \xi_i$ and random variables with delay in zero.

$$Y_{n+1} = \max(0, Y_n + \xi_{n+1}).$$

Really, at first consider a special case $\sigma^2 = D\xi_1 < \infty$. Prove that the sequence $\frac{X_n - n\mu}{\sigma\sqrt{n}}, n \ge 1$ is uniformly continuous in probability. Without loss of generality, assume $\mu = E\xi_1 = 0$ and $\sigma = 1$.

Denote $$X^*_n = \frac{X_n}{\sqrt{n}} = \frac{\xi_1 + \xi_2 \ldots + \xi_n}{\sqrt{n}}.$$

For $k, n \ge 1$ we have

$$|X^*_{n+k} - X^*_n| \le \frac{1}{\sqrt{n}}|X_{n+k} - X_n| + |X^*_n|\left(1 - \left(\frac{n}{n+k}\right)^{1/2}\right).$$

Hence, when $\varepsilon > 0, \delta > 0$ for $k \le n\delta$ we get

$$P\left\{\max_{k\le n\delta} |X^*_{n+1} - X^*_n| > \varepsilon\right\} \le P\left\{\max_{k\le n\delta} |X_{n+k} - X_n| \ge \frac{\varepsilon/2\sqrt{n}}{2}\right\} + P\left\{|X^*_n| > K(\delta)\frac{\varepsilon}{2}\right\},$$

Where $K(\delta) = \left(1 - (1+\delta)^{-1/2}\right)^{-1}$.

It is clear that the sequence $X^*_n, n \ge 1$ is stochastically bounded and therefore as $\delta \to 0$

$$P\left\{|X^*_n| > K(\delta)\frac{\varepsilon}{2}\right\} \to 0$$

For all $n \ge 1$ since $K(\delta) \to \infty$ as $\delta \to 0$.

Further, by the Kolmogorov inequality we have

$$P\left\{\max_{k\le n\delta} |X_{n+k} - X_n| \ge \frac{\varepsilon}{2}\sqrt{n}\right\} \le \frac{4}{\varepsilon^2}\delta\sigma^2.$$

Consequently, by Definition 2 the sequence $X^*_n, n \ge 1$ is uniformly continuous in probability. Similar calculations show that the sequence $Y^*_n = \frac{Y_n - n\mu}{\sigma\sqrt{n}}, n \ge 1$ is also uniforinly continuous by probability [3].

In the general case $1 < \alpha \leq 2$, fulfillment of conditions of uniform continuity in probability for the indicated chains follow from the papers [3,11].

Remark 2. Notice that conditions of convergence $X_n \to \infty$, almost surely appearing in all three theorems, is equivalent to irrevocability of irreducible Markov chain from the set $Z^+ = \{0,1,2,...\}$ [6]. We also notice that statements of Theorems 1,2 are valid for homogeneous in time and partially homogeneous in space Markov chains whose trajectory behaves as an ordinary process of summation of identically distributed random variables. The examples of homogeneous in time and partially homogeneous in space Markov chains give the chains from Remark 1 and also oscillating walks whose distribution of the jump coincides with distribution of some random variables ξ and η, respectively, in domains $\upsilon > 0$ and $\upsilon < 0$ i.e. $\xi_n(\upsilon) \overset{d}{=} \xi$ for $\upsilon > 0$ and $\xi_n(\upsilon) \overset{d}{=} \eta$ for $\upsilon < 0$ (see [3]).

III. AUXILIARY FACTS

For proving Theorem 1 we'll need the following facts formulated in the form of lemmas.

Lemma 1. Let the Markov chain X have an asymptotically homogeneous in time and space a mean drift $\mu > 0, X_n \to \infty$ almost surely as $n \to \infty$ and for some spatial level U and time N the family of random variables $\{\xi_n(\upsilon), n \geq N, \upsilon \geq U\}$ have an integrals majoring.
Then

1) $P(\tau_c < \infty) = 1$ for all $c \geq 0$;
2) $\tau_c \to \infty$ as $c \to \infty$;
3) $\frac{\tau_c}{c} \to \frac{1}{\mu}$ as $c \to \infty$;

These statements were proved in paper [15].

Lemma 2. (Anscombe's theorem). Let $\tau_c, c > 0$ be a family of non negative integer random variables such that $\frac{\tau_c}{c} \xrightarrow{P} a > 0$ as $c \to \infty$ and the sequence of random variables $Y_n, n \geq 1$ be uniformly continuous in probability.

$$Y_{t_c} - Y_{[c\alpha]} \xrightarrow{P} 0 \text{ as } c \to \infty$$

Then if Y_n weakly converges to the random variable Y, then Y_{t_c} also weakly converges to Y as $c \to \infty$.
The proof of Anscombe's theorem may be found in [14,16].

Lemma 3. If the sequences X_n and $Y_n, n \geq 1$ are uniformly continuous in probability, then the sequence $X_n + Y_n, n \geq 1$ is also uniformly continuous in probability and furthermore, if X_n and $Y_n, n \geq 1$ are stochastically bounded and the function K is continuous on R^2, then the sequence $K(X_n, Y_n), n \geq 1$ is uniformly continuous in probability.
The proof of this lemma is given in [16].

IV. PROOF OF BASIC RESULTS

Proof of Theorem 1. Denote by $X_c = X_{\tau_c} - c$ the overshoot of the Markov chain X for the boundary c.
We have

$$\frac{X\tau_c - \mu\tau_c}{\sigma\sqrt{\tau_c}} = \frac{c - \mu\tau_c}{\sigma\sqrt{\tau_c}} + \frac{X_c}{\sigma\sqrt{\tau_c}}. \qquad (3)$$

In the conditions of the proved theorem these is a statement of theorem 5 from the paper (Korshunov 2001), by which

$$\lim_{n\to\infty} P\left(\frac{X_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \Phi(x),\ x \in R, \qquad (4)$$

Then, according to **Lemma 1** and the Anscombe theorem, by the conditions of uniform continuity in probability of the sequence $\frac{X_n - n\mu}{\sigma\sqrt{n}}, n \ge 1$ we get

$$\lim_{c\to\infty} P\left(\frac{X_{\tau_c} - \mu\tau_c}{\sigma\sqrt{\tau_c}} \le x\right) = \Phi(x),\ x \in R.$$

Now, from equalities (3) and (4) for obtaining the statement of **Theorem 1** it is enough to show that

$$\frac{X_c}{\sqrt{\tau_c}} \xrightarrow{P} 0 \text{ for } c \to \infty \quad (5)$$

Indeed, we have

$$\frac{X_c}{\sqrt{\tau_c}} = \frac{X_{\tau_c} - c}{\sqrt{\tau_c}} \le \frac{X\tau_c - X_{\tau_c - 1}}{\sqrt{\tau_c}} \qquad (6)$$

By **Lemma 3**, the sequence

$$\frac{X_n - X_{n-1}}{\sqrt{n}} - \frac{X_n - n\mu}{\sqrt{n}} = \frac{X_{n-1} - (n-1)\mu}{\sqrt{n}} + \frac{\mu}{\sqrt{n}}, n \ge 1$$

is uniformly continuous in probability.

Consequently for each fixed state v of the Markov chain X the sequence $\frac{\xi_n(v)}{\sqrt{n}}, n \ge 1$ is uniformly continuous in probability. It follows from the condition of uniform integrability of the family of squares of the jumps of the chain X that the family $|\xi_n(v)|, n \ge N, \upsilon \ge V$ possesses an integrable majorant.

Therefore,

$$\frac{\xi_n(\upsilon)}{\sqrt{n}} \xrightarrow{P} 0 \text{ as } n \to \infty$$

uniformly with respect to $\upsilon \ge V$.

Then, by **Lemma 2**, $\frac{\xi_{\tau_c}(\upsilon)}{\sqrt{\tau_c}} \xrightarrow{P} 0$ as $c \to \infty$ uniformly with respect to $\upsilon \ge U$.

Hence, taking into account that distribution of the difference $X_{\tau_c} - X_{\tau_c - 1}$ coincides with distribution of the jump of the chain X at time τ_c, from (6) we get (5).

Now, from (3) and (5) we find

$$\lim_{c\to\infty} P\left(\frac{c - \mu\tau_c}{\sigma\sqrt{\tau_c}} \le x\right) = \Phi(x),\ x \in R.$$

By statement 3) of **Lemma 1**, we easily get the statement of **Theorem 1** from the last relation.

Remark 3. The condition $\lim_{\delta\to 0}\lim_{n\to\infty} \frac{A([n(1+\delta)])}{A(n)} = 1$ was introduced in the paper [11]. It provides fulfilment of the condition of uniform continuity in probability of the sequence of the normalized sums

$$\frac{\sum_{k=1}^{n} \xi_k - E\left(\sum_{k=1}^{n} \xi_k\right)}{A(n)}$$

of independent, generally speaking, differently distributed random variables $\xi_n, n \geq 1$. In the case of independent identically distributed random variables, one can take $A(n) = n^{1/\alpha} L(n), 0 < \alpha \leq 2$, where $L(x), x > 0$ is a slowly changing function at infnity [12] and it is easy to verify that in this case the given condition is fulfilled.

Proof of Theorem 2. It follows from definition of the quantity τ_c and the condition of monotonicity of the Markov chain X_n that

$$P(\tau_c > n) = P\left(\max_{0 \leq k \leq n} X_k \leq c\right) = P(X_n \leq c).$$

Hence we have

$$P(\tau_c > n) = P\left(\frac{X_n}{A(n)} \leq \frac{c}{A(n)}\right). \qquad (7)$$

Let $c \to \infty$ and $n = n(c) \to \infty$, so that $\frac{c}{A(n)} = x > 0$.

It follows from the well known properties of slowly changing functions that the solution of the equation

$\frac{c}{A(n)} = x$ with respect to n has the asymptotics (see [12]):

$$n \sim \left(\frac{c}{x}\right)^{\alpha} L_{1/\alpha}^{*}(c), c \to \infty .$$

By this asymptotics, the statement of **Theorem 2** follows from (7).

Remark 4. In the statement of **Theorem 2**, when the distribution of the jump of the Markov chain belongs to normal attraction domain of the stable law, one can take $A(n) = \sigma n^{1/\alpha}$ and $L_{1/\alpha}^{*}(c) \equiv \sigma^{-\alpha}$, where $0 < \alpha < 1$ and $\sigma > 0$ is some constant [2].

V. APPLICATION

In this section we consider the following model from the theory of branching processes. Let at the initial moment , at the point $x \in g = (0, l), 0 < l < \infty$ there be n particles of type T_1. These particles are subjected to diffusion, reproduce and generate the particles of type T_2 and absorbed on the boundary (at the points $x = 0$ and $x = l$), but the particles of type T_2 don't die, don't reproduce and don't absorb on the boundary. Thus, the particles of type T_2 is a finite product of diffusion.

Denote by S_n the number of particles of type T_2 originated from n particles of type T_1, and by ξ_k the number of particles of type T_2 obtained from the k -th particle (in any enumeration) of type T_1 available at initial moment at the point $x \in R$.

Then, we can represent S_n in the from of the sum of n independent identically distributed random variables $\xi_1, \xi_2, \ldots, \xi_n : S_n = \sum_{k=1}^{n} \xi_k$.

Consider the first passage moment

$$t_c = \inf \{n \geq 1 : S_n > c\}$$

of the sum S_n for the level $c \geq 0$.

The first passage moment t_c arises in many applied problems that may be solved by this or other model of random walks described by approximately branching process. For example, the quantity t_c may play a role of time of the first propagation of the epidemic process (when the number of patients for the first time exceeds for critical level) that with very good approximation may be assumed a branching process [13].

It is shown in [13] that the distribution of the random variable ξ_k belongs to stable law with parameters $\alpha = 1/2$, $\beta = 1$, namely,as $n \to \infty$

$$\lim_{n\to\infty} P\left(\frac{S_n}{L^2\varphi^2(x)n^2} \le y \right) \to G_{1/2}(t) = \frac{1}{2\sqrt{\pi}}\int_0^y z^{-3/2}e^{-1/4y}dz , \qquad (8)$$

where the constants L and $\varphi(x)$ are the characteristics of the generating function, of the number of the particles in generations whose obvious forms are reduced in [13,Theorem 3]). Applying Theorem 2 to the sum S_n, from relation (8) we can find limit distribution for t_c as $c \to \infty$.

For that it is enough to take in Theorem 2 (see also Remark 4) $\alpha = 1/2, \sigma = L^2\varphi^2(x)$ and $L_2^*(c) = \dfrac{1}{L\varphi(x)}$. Then we get

$$\lim_{c\to\infty} P\left(\frac{t_c}{L\varphi(x)\sqrt{c}} > y^{-1/2} \right) \to G_{1/2}(y),\ t > 0 .$$

REFERENCES

[1] Chenq-Der Fuh., Asymptotic operating characteristics of an optimal detection in hidden Markov models.The Annals of statistics vol. 32. No. 5, 2004, pp. 2305-2339.

[2] V. Feller, Introduction to the theory of probabities and its applications, vol 2, M., 1984, p. 750.

[3] B. Jurgen, K. Gotz, A random walk approach to Calton-Wotson trees. Journal of Theoretical Probability, vol. 13, No. 3, 2000, pp. 777-792.

[4] V. Melfi, Nonlinear Markov renewal theory with statistical applications, Ann. Probab., vol.20, No.2, 1992, pp. 753-771.

[5] J.Mogyoradi, On limiting distributions for sums of a random of independent random variables-Mogyar tud. Akad. Math. Kulato int. kozl. A6, No 3, 1961, pp. 365-371.

[6] E.Seneta, Regular varying functions-Lect. Notes math., 1976, B. 508.

[7] D. Siegmund,Sequantial analysis. Tests and confidence intervals. New York. Etc. springer-Verlag, 1985, 272 p.

[8] M. Woodroofe, Nonlinear renewal theory in sequential analysis. SIAM. 1982, 110 p

[9] D.Korshunov, Limit theorems for general Markov chairs. Sib. matem. journal, vol. 42, 2001, pp. 354-371.

[10] V. Afanasyev, Galton-Watson process under condition of achievement of high level, Teoria veroyat. i ee primen., vol.52, issue 3, 2007, pp.588-594.

[11] S. Aliyev, T. Hashimova, Asymptotic behavior of the conditional probability of the nonlinear boundary crossing by a random walk. - Theory of Stochastic Processes 16 (32), No 1, 2010, pp. 12-17

[12] A. Borovkov, Asymptotics of probability of boundary crossing by the Markov chain trajectory. Exponentially decaying tails of jumps. Teoria veroyat. i ee primen. vol. 48, issue 2, 2003, pp. 254-273.

[13] V. Sevestyanov, Branching processes. M.: 1971, 436 p.

[14] I. Gikhman, A.V. Skorokhod. Theory of random processes. Vol. 1, M., 1971, 664 p.

[15] F. Rahimov, V.Abdurakhmanov,On limit behavior of linear first passage time of the Markov chain.-Proceedings of IMM of NAS of Azerbaijan, XXVII, 2007, pp. 69-74.

[16] V. Kalashnikov, Quality analysis of behavior of complicated systems by test function method. M., 1978, 247 p.