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THE SIMULTANEOUS DUAL INTEGRAL EQUATIONS WITH THE HELP OF KERNEL OF FOX

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Abstract— In this paper the formal solution of certain simultaneous dual integral equations involving H-functions is obtained by the method of fractional integration. By the application of fractional integration operators, the given simultaneous equations are transformed into two others having common kernel and the problem then reduces to that of solving one integral equation. Since the common kernel obtained is a symmetrical Fourier kernel given earlier by Fox the solution then follows easily.

Keyword—Dual integral equations, Kernel of Fox, H-function, fractional Integration etc

I. INTRODUCTION

Fox [3] and Saxena [6] have obtained the formal solution of certain dual integral equations involving H-functions. In this section we have discussed certain simultaneous dual integral equations of more general nature associated with the kernel of Fox [3] **.** By the application of fractional integration operators the given integral equations are transformed into two others with common kernel and problem then reduces to that of solving one integral equation. Since the common kernel is the same as investigated earlier by Fox, the formal solution is readily obtained.

We define the H-function by means of Mellin Barnes integral [1] , in the form

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,A_1)\ldots\ldots(a_p,A_p)\\(b_1,B_1)\ldots\ldots(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\left(b_j+B_js\right)!\,\prod_{i=1}^{n}(1-a_i-A_is)!}{\prod_{j=m+1}^{p}(1-b_j-B_js)!\,\prod_{i=n+1}^{q}(a_i+A_is)!}x^{-S}ds, \tag{1.1}$$

Where x is not equal to zero and an empty product is to be interpreted as unity; p, q, m and n are integers, satisfying $1\le m\le q,\ 0\le n\le 1$; A_i and B_j are positive numbers and a_i and b_j are complex numbers (where $i=1,2,\ldots\ldots,p$ and $j=1,2,\ldots\ldots,q$) such that no pole of $(b_j+B_{j^s})!\,(j=1,\ 2,\ldots\ldots,m)$ coincides with any pole of $(1-B_i-sA_i)!\,(i=1,2,\ldots\ldots,n)i.e.$

$$A_i\left(b_j+v\right)\neq B_j(a_i-\eta-1)\ \text{ for } v,\eta=0,1,2,\ldots\ldots$$

$$H_{2m,2n}^{n,m}\left[x\left|\begin{matrix}(1-a_m,A_m),(a_m-A_m,A_m)\\(b_n,B_n),(1-b_n-B_n,B_n)\end{matrix}\right.\right]=H_{2m,2n}^{n,m}(x)=\frac{1}{2\pi i}\int_L \chi_{m,n}(s)x^{-s}\,ds, \tag{1.2}$$

Where

$$\chi_{m,n}(s)=\prod_{i=1}^{n}\frac{(b_i+sB_i)!}{(b_i+B_i-sB_i)!}\ \prod_{j=1}^{m}\frac{(a_j+sA_j)!}{(a_j-A_j+sA_i)!} \tag{1.3}$$

behaves as a symmetrical kernel. From (1.2) it follows that the Mellin transform of $H_{2m,2n}^{n,m}(x)$ is

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$\chi_{m,n}(s)$.

II. SIMULTANEOUS DUAL INTEGRAL EQUATIONS

The solution of the following simultaneous dual integral will be obtained here:

$$\int_0^\infty H_{2m,2n}^{n,m}\left[xu\left|\begin{matrix}(1-\alpha_m^k,A_m),\ (a_m-A_m,A_m)\\(b_n,B_n),\ (1-\beta_n^k-B_n,B_n)\end{matrix}\right.\right]\sum_{h=1}^n a_{hk}f_h(u)du=\varphi_k(x),\qquad 0<x<1\,, \tag{1.4}$$

and

$$\int_0^\infty H_{2m,2n}^{n,m}\left[xu\left|\begin{matrix}(1-a_m,A_m),\ (\alpha_m^k-A_m,A_m)\\(\beta_n^k,B_n),\ (1-b_n-B_n,B_n)\end{matrix}\right.\right]\sum_{h=1}^n b_{hk}f_h(u)du=\psi_k(x),\qquad x>1\,, \tag{1.5}$$

Where $k=1,2,\dots\dots,n$; $\varphi_k(x)$ and $\psi_k(x)$ are given, $f_h(u)$ are to be determined and a_{hk} , b_{hk} are known constants. The H-functions used here are as defined in (1.2). In above equations (1.4) and (1.5) the unknown function $f_h(u)$ must be bounded and integrable.

If we put $M[f(u)]=F(s)$ and apply Fox's result [4] to (1.4) and (1.5), we obtain

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)\,x^{-s}\sum_{h=1}^n a_{hk}\,F_h(1-s)\,ds=\varphi_k(x),\quad 0<x<1, \tag{1.6}$$

and

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)\,x^{-s}\sum_{h=1}^n b_{hk}\,F_h(1-s)\,ds=\psi_k(x),\quad x>1, \tag{1.7}$$

Where $k=1,2,\dots\dots n$, and

$$\chi_{m,n}^{*}(s)=\prod_{i=1}^n\frac{(b_i+sB_i)!}{(\beta_i^k+B_i-sB_i)!}\ \prod_{j=1}^m\frac{(\alpha_j^k-sA_j)!}{(a_j-A_j+sA_j)!}\,, \tag{1.8}$$

$$\chi_{\mathrm{m,n}}^{**}(s)=\prod_{i=1}^n\frac{(\beta_i^k+sB_i)!}{(b_i+B_i-sB_i)!}\ \prod_{j=1}^m\frac{(a_j-sA_j)!}{(\alpha_j^k-A_j+sA_j)!}\,,$$

(1.9)

III. REDUCTION OF (1.6) AND (1.7) TO EQUATIONS WITH A COMMON KERNEL

The well known beta function can be expressed as

$$\int_0^x v^{C_m\,a_m-S-1}\,(x^{C_m}-v^{C_m})^{(\alpha_m-a_m-S-1)}\,dv=\frac{(\alpha_m-a_m)!(a_m-sA_m)!}{C^m\,(\alpha_m-sA_m)!}\,x^{C_m\alpha_m-C_m-S}\,, \tag{1.10}$$

Where $c_m=\frac{1}{A_m}$, $\alpha_m>a_m$ and $c_ma_m=\frac{a_m}{A_m}>\sigma_0$, where $s=\sigma_0+it$ on the line $\sigma_0=0$.

When fractional integration operators are introduced some of these conditions may no longer be necessary and the second may be relaxed.

Replacing x by v in (1.6), multiplying by $v^{C_ma_m-1}(x^{C_m}-v^{C_m})(\alpha_m^k-a_m-1)$ and integrating (which is justified under the assumptions already made) through the integral sign with respect to v from $0\ to\ x, 0<x<1$, we find that

$$\frac{1}{2\pi i}\int_L \chi_{m,n}^{(1)}(s)x^{-s}\sum_{h=1}^n a_{hk}F_h(1-s)\,ds$$

$$=\frac{C_mx^{C_m-\alpha_n^mC_m}}{(\alpha_m^k-a_m)!}\int_0^x v^{C_ma_m-1}(x^{C_m}-v^{C_m})^{\alpha_m^k-a_m-1}\,\varphi_k(v)dv, \tag{1.11}$$

Where $\chi_{m,n}^{(1)}(s)$ is obtainable from $\chi_{m,n}^{*}(s)$ by replacing β_m^k by a_m only.

The operator of fractional integration denoted by R is used in the form

$$R[\lambda,\delta\colon m\colon w(x)] = \frac{m}{\omega!}\, x^{-\delta+m\lambda+m-1} \int_0^x (x^m - v^m)^{\lambda-1}\, v^{\delta}\, \omega(v) dv. \quad (1.12)$$

The case $m = 1$ is due to Köber [5] and the result for more general case $m > 0$ has been given by Erdelyi [2]. Fox [3] has shown that there is no essential difference between the two cases, since

$$R[\lambda,\delta\colon m\colon w(x)] = R[\lambda, m^{-1}(\delta+1)^{-1} - 1\colon 1\, w(x)], where\ x^m = X,\ v^m = V\ and\ w(x) = w(x).$$
(1.13)

The operator $R(x)$ exists provided $(x)\, \varepsilon L_p(0,\infty)\; p \geq 1, \gamma > 0, \delta > \frac{1}{q}$. If in addition, $w(x)$ can be differentiated sufficiently often, then the operator R exists for negative as well as positive values of γ.

For brevity, we write

$$R\big[(\alpha_i^k - a_i), (a_i {A_i}^{-1} - 1), {A_i}^{-1} : w(x)\big] = R_i[w(x)] \quad (1.14)$$

$$R\big[(b_j - \beta_j^k), (\beta_j^k + B_j), {B_j}^{-1} - 1 : {B_j}^{-1} w(x)\big] = R_j^*[w(x)]. \quad (1.15)$$

From (1.14) it is seen that right hand side of (1.11) is equal to $R_m[\varphi_k(x)]$ with $0 < x < 1$. On transforming the equation (1.6) step by step by means of the operators R_i and R_j^* for $i = m, (m-1) \ldots \ldots, 2, 1$, and $j = n, n-1, \ldots \ldots .2, 1$, we find that

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)\, x^{-s} \sum_{h=1}^{n} a_{hk}\, F_h(1-s)\, ds$$
$$= R_1^*[R_2^* \ldots R_n^*, R_1 \ldots R_m[\varphi_k(x)] \ldots], \quad 0 < x < 1.$$
(1.16)

We now proceed to transform the integral equation (1.7), again form beta function formula, we have

$$\int_x^\infty v^{d_n - d_n\beta_n^k - S - 1}(v^{d_n} - x^{d_n})^{\beta_n^k - b_n - 1}\, dv = \frac{(\beta_n^k - b_n)!(b_n + sB_n)!}{d_n(\beta_n^k + sB_n)!}\, x^{-s-d_n b_n}, \quad (1.17)$$

Where $d_n = B_n^{-1}, \beta_n^k > b_n$ and $d_n\sigma_0 + b_n > 0$.

In (1.7) let us replace $x\ by\ v$ multiply by $v^{d_n - d_n\beta_n^k - 1}(v^{d_n} - x^{d_n})^{\beta_n^k - b_n - 1}$ and integrate with respect to form $x\ to\ \infty$ through the integral sign. Equation (1.7) takes the form

$$\frac{1}{2\pi i}\int_L \chi_{m,n}^{(2)}(s) x^{-s} \sum_{h=1}^{n} b_{hk} F_h(1-s)\, ds$$
$$= \frac{d_n x^{d_n b_n}}{(\beta_n^k - b_n)!} \int_x^\infty v^{d_n - \beta_n^k d_n - 1}(v^{d_n} - x^{d_n})^{\beta_n^k - b_n - 1}\, \varphi_k(v) dv, \quad (1.18)$$

Where $x > 1, d_n = B_n^{-1}$ and $\chi_{m,n}^{(2)}(s)$ is obtainable from $\chi_{m,n}^{**}(s)$ by the replacement of β_n^k by b_n .

The second operator of fractional integration denoted by K required here is

$$K[\gamma, \delta : n :\ \omega(x)] = \frac{n}{\sqrt{\gamma}}\, x^{\delta} \int_x^\infty (v^n - x^n)^{\gamma-1} v^{-\delta-n\gamma+n-1}\, \omega(v)\, dv\,. \quad (1.19)$$

If $\omega(x)\varepsilon L_p(0,\infty), p > 1$ and $\omega(x)$ can be differentiated sufficiently often K exists, provided $n > 0, \delta > \frac{1}{p}, where\ \gamma$ can have any positive or negative values.

In the contracted notations, we write

$$K\left[\left(\beta_i^k - b_i\right), b_i B_i{}^{-1} B_i : \omega(x)\right] = K_i[\omega(x)], \text{ and} \tag{1.20}$$

$$K\left[\left(a_j - a_j^k\right), \left(\alpha_j^k - A_j\right) A_j{}^{-1}; A_j^{-1} : \omega(x)\right] = K_j^*[\omega(x)]. \tag{1.21}$$

From (1.20) it is evident that the right hand side of (1.18) is $K_n[\psi_k(x)]$, where $x > 1$.

The successive application of the operators K_i for $i = n, n-1, \dots\dots, 2, 1$; and K_j^* for $j = m, m-1, \dots\dots 2, 1$ to (1.7), yields the integral equation

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)x^{-s} \sum_{h=1}^{n} a_{hk} F_h(1-s)\ ds = \sum_{h=1}^{n} c_{hk}\ K_1^*[K_2^* \dots K_m^*\ K_1\ K_2 \dots K_n[\psi_k(x)] \dots], \tag{1.22}$$

Where c_{hk} are the elements of the matrix $[a_{hk}][b_{hk}]^{-1}$.

If we set $g_k(x) = \begin{cases} R_1^*[R_2^* \dots R_n^*, R_1 R_2 \dots R_m[\varphi_k(x)] \dots], & 0 < x < 1, \\ \sum_{h=1}^{n} c_{hk} K_1^*[K_2^* \dots K_n^*\ K_1\ K_2 \dots\ K_m[\psi_k(x)] \dots], & x > 1, \end{cases}$

$k = 1, 2, \dots\dots, n.$
(1.23)

Equations (1.16) and (1.22) can be put into the compact form as

$$\frac{1}{2\pi i}\int_L \chi_{m,n}(s)\ x^{-s} \sum_{h=1}^{n} a_{hk}\ F_h(1-s)\ ds = \frac{1}{2\pi i}\int_L \mathrm{M}\left[\mathrm{H}_{2m,2n}^{n,m}(u)\right] x^{-S} \sum_{h=1}^{n} a_{hk}\ F_h(1-s) ds$$

$$= g_k(x)$$

Where M denotes the Mellin transform. Since $H_{2m,2n}^{n,m}(xu)$ is a symmetrical Fourier kernel, the formal solution can be readily obtained as

$$f_h(x) = \sum_{h=1}^{n} d_{hk} \int_0^\infty H_{2m,2n}^{n,m}\left[(xu)\left|\begin{matrix}(1-a_m, A_m), (a_m - A_m, A_m)\\(b_n, B_n), (1-b_n - B_n, B_n)\end{matrix}\right.\right] g_k(u) du$$

$$= \sum_{h=1}^{n} d_{hk} \int_0^1 H_{2m,2n}^{n,m}\left[(xu)\left|\begin{matrix}(1-a_m, A_m), (a_m - A_m, A_m)\\(b_n, B_n), (1-b_n - B_n, B_n)\end{matrix}\right.\right]\{R_1^*[R_2^* \dots R_n^*, R_1 R_2 \dots R_m[\varphi_k(x)] \dots]\}$$

$$+ \int_1^\infty H_{2m,2n}^{n,m}\left[(xu)\left|\begin{matrix}(1-a_m, A_m), (a_m - A_m, A_m)\\(b_n, B_n), (1-b_n - B_n, B_n)\end{matrix}\right.\right] \sum_{h=1}^{n} c_{hk} K_1^*[K_2^* \dots K_n^*\ K_1\ K_2 \dots\ K_m[\psi_k(x)] \dots]$$

Where $h = 1, 2, \dots\dots, n$.
(1.25)

IV. INTERESTING PARTICULAR CASES

(i) If we set $m = 0,\ n = 1, B_1 = B_2 = 1, b_1 = \lambda^k + \frac{V^k}{2}, \beta_1^k = \frac{V^k}{2} - \lambda^k$ and use the identity[2].

$$G_{0,2}^{1,0}\left(x \middle| a, b\right) = x^{\frac{a+b}{2}} J_{a-b}\left(2\sqrt{x}\right), \tag{1.26}$$

We find that the formal solution of the simultaneous dual integral equations

$$\int_0^{\infty}(ux)^{\lambda^k} J_{v^k}\left[2\sqrt{(xu)}\right] \sum_{h=1}^{n} a_{hk}\, F_h(u) = \varphi_k(k), \quad 0 < x < 1, \tag{1.27}$$

$$\int_0^{\infty}(ux)^{-\lambda^k} J_{v^k}\left[2\sqrt{(xu)}\right] \sum_{h=1}^{n} a_{hk}\, F_h(u) = \psi_k(k), \qquad x > 1; \tag{1.28}$$

$k = 1, 2, \ldots\ldots, n\,;$

is given by

$$f_h(x) = \sum_{h=1}^{n} d_{hk} \int_0^1 J_{v^k+2\lambda^k}\left(2\sqrt{(xu)}\right)\, R\left[2\lambda^k,\ \frac{v^k}{2} - \lambda^k, 1, \varphi_k(u)\right] du$$

$$+ \int_1^{\infty} J_{v^k+2\lambda^k}\left(2\sqrt{(xu)}\right) \sum_{h=1}^{n} c_{hk}\, K\left[-2\lambda^k,\ \lambda^k + \frac{v^k}{2}, 1, \psi_k(u)\right] du, \quad h = 1,\ 2, \ldots\ldots, n \tag{1.29}$$

(ii) On the other hand if we put $m = 1, n = 0, A_1 = A_2 = 1, \alpha_1^k = 1 - P^k + \frac{\mu^k}{2}$,

$a_1 = 1 + P^k + \frac{\mu^k}{2}$ and apply the transformation [2].

$$G_{p,q}^{m,n}\left[x \left|{}^{a_p}_{b_p}\right.\right] = G_{q,p}^{n,m}\left[x^{-1} \left|{}^{1-b_q}_{1-a_p}\right.\right], \tag{1.30}$$

$$G_{0,2}^{1,0}\left(x \,\middle|\, a, b\right) = x^{\frac{a+b}{2}} J_{a-b}\left(2\sqrt{x}\right), \tag{1.31}$$

It is found that the formal solution of the simultaneous dual integral equations

$$\int_0^{\infty}(xu)^{p^k-1} J_{v^k}\left[2(xu)^{-1/2}\right] \sum_{h=1}^{n} a_{hk}\, f_h(u)du = \varphi_k(x), \qquad 0 < x < 1, \tag{1.32}$$

$$\int_0^{\infty}(xu)^{p^k-1} J_{v^k}\left[2(xu)^{-1/2}\right] \sum_{h=1}^{n} b_{hk}\, f_h(u)du = \psi_k(x), \qquad x > 1, \tag{1.33}$$

is given by

$$f_h(x) = \sum_{h=1}^{n} d_{hk}\left[\int_0^1 (xu)^{-1} J_{v^k+2p^k}\ \left(2(xu)^{-1/2}\right) R\left[-2P^k,\ P^k + \frac{\mu^k}{2}, 1, \varphi_k(u)\right]\right] du$$

$$+ \int_1^{\infty}(xu)^{-1} J_{v^k+2p^k}\left(2(xu)^{-1/2}\right) \sum_{h=1}^{n} c_{hk}\, K\left[2p^k,\ 2\mu^k - p^k,\ 1,\ \psi_k(u)\right] du$$

Where $h = 1, 2, \ldots\ldots, n$.

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