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SOME QUANTITATIVE ANALYSIS OF COMPUTATIONAL METHODS APPLIED TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract-The aim of the present dissertation is to make a quantitative comparison of computed of computed solution of some ordinary differential equations by different numerical techniques and to draw out certain observations with some fundamental results from linear algebra, theory of differential equations and numerical solution of differential equations.

CHAPTER-1

1.1 INTRODUCTION

In most of modern physical situations we need to solve a set of differential equations subject to some initial conditions and/or boundary conditions in the areas, particularly in mathematical physics and mathematical biology we will face partial differential equations, integral differential equations, difference equations and differential equations of even more complex type.

Determining the deflection of simply supported beam where the deflection and derivative at the and points are specified is a typical example of boundary value problems. The heat flow problem in general fall in the boundary value problem because the temperature and temperature gradients are given at the two ends. The vibrating strings membranes and flow of fluids through tubes are some examples which involves boundary value problems.

The procedure for solving boundary problems in partial differential equations very much demand the procedure employed for solving ordinary differential equations with boundary conditions, may be a Laplace transform method or separation of variables method etc. Hence the study of ordinary differential equations is the basis for study of partial differential equations.

Here we consider the numerical study of ordinary differential equations with two point boundary values. Some three examples of solving two point boundary value problems have been considered for this study. Quasilinearization technique. Shooting method, finite difference method and finite element method are employed while working out the solutions of these examples.

1.2. CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves differential coefficients. The order of a differential equation is the order of highest derivative appearing in it. A differential equation will have unique solution when it is subject to as many conditions as the order of the equation. If the conditions mentioned on a differential equation are less than the order of the equation the equation will have infinite solutions represented by K parameter family of curves where K= order – number of conditions. If the conditions on a differential equation are more than the order the equation the equation may not have a solution. A differential equation is called well posed if the equation is given with as many conditions as order of the equation. Otherwise the problem is said to be ill posed. A differential equation is called linear if the unknown and its derivatives appeared only once in each term with degree one. Otherwise the equation is called non-linear. For example a second order linear equation will be of the form

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$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x)..........(1.2.1)$$

This equation is called homogeneous if the right hand side function f(x) is zero. Otherwise it is called nonhomogeneous. The conditions on these equations are of the form

$$\left.\begin{aligned} \alpha_1 y(a) + \beta_1 \frac{dy(a)}{dx} = \gamma_1 \\ \alpha_2 y(b) + \beta_2 \frac{dy(b)}{dx} = \gamma_2 \end{aligned}\right\} \qquad ......(1.2.2)$$

If $\gamma's$ are zero the conditions are called homogeneous boundary conditions.

If Y(a) = γ_1 y(a) = γ_2

hen the conditions are called initial conditions.

If the conditions are mentioned in general form as

$$g[y(a), y(b), y(a), y(b)] = 0 \qquad .....(1.2.3.)$$

the conditions are called nonlinear.

A first order linear equation is of the form

$$\frac{dy}{dx} + P(x)y = f(x), y(o) = a_1 \qquad ......(1.2.4.)$$

the solution of this equation is given by

y(x) = exp

A first order nonlinear differential equation is of the form

$$\frac{dy}{dx} = f(x,y) \qquad \boldsymbol{y(O) = a_1} \qquad ....(1.2.6)$$

Any linear equation of order n can be split into n first order equations and this set of n equations can be put in matrix form as

$$\frac{d\bar{y}}{dx} = A\bar{y}$$

Where

$$\bar{y} = \begin{bmatrix} y = y_1 \\ y = y_2 \\ y^{(n)} + y_n \end{bmatrix} \qquad \text{and } A = \begin{bmatrix} 0 & 1 & 0....... & 0 \\ 0 & 0 & 1..... & 0 \\ 0 & 0 & 0...... & \\ ^-P_n & ^-P_{n-1} & .... & P_1 \end{bmatrix}$$

.....(1.2.7)

Where P_1, P_2........P_n are coefficients of y^{n-1},y^{n-2}, ...y.

The general form of second order nonlinear equation with non homogeneous boundary conditions is of the form

y = f(x,y,y') with

$y(x_o)=a_1$ $y(x_n)=a_2$ .....(1.2.8)

here the aim is at the solution of second order differential by various numerical methods.

1.3. SOLUTION OF TRIDIAGONAL SYSTEM OF EQUATIONS:

Matrices occur in a variety of problems of interest; for example in solution of linear algebraic and eigen value problems. The matrix notation is convenient and powerful in expressing basic relationship in fields like elasticity and electrical engineering. While solving boundary value problems the in finite difference method or in finite element method tri diagonal system of equations are extracted.

Consider the system of equation defined by

$b_1 u_1 + c_1 u_2 = d_1$
$a_2 u_1 + b_2 u_2 + c_2 u_9 = d_2$
$a_3 u_2 + b_3 u_{93} + c_3 u_4 = d_3$
. . . .
$a_n u_{n-1} + b_n u_n = d_n$ . . . . . (1. 3. 1.)

The matrix coefficient is

$$A = \begin{matrix} b_1\, c_1\, 0\, 0\, 0\, 0 \ldots\ldots 0 \\ A_2\, b_2\, c_2\, 0\, 0\, 0 \ldots\ldots 0 \\ 0\, a_3\, b_3\, c_3\, 0\, 0 \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots \\ 0\, 0\, 0 \ldots a_{n-1}\, b_{n-1}\, c_{n-1} \\ 0\, 0 \ldots\ldots\ldots\ldots a_n\, b_n \end{matrix}$$

Matrix of type (1. 3. 2) is called tri diagonal matrix which occur frequently in the solution of dinary differential equations by finite element method of finite difference method.

The method of factorization can be conveniently applied to solve the system (1. 3. 1) using computational procedure given by Thomas. This procedure is given in detail in foregoing chapter 3(3. 1a).

CHAPTER-2

In this chapter we briefly outline some of numerical techniques employed in computing the solutions of two point boundary value problems.

1) Quasilineraization technique
2) Shooting method (SHM)
3) Finite difference method (FDM)
4) Finite element method (FEM)

2.1 <u>QUASTILINEARIZATION</u>

We now turn out attention to the study of nonlinear second order differential equation of the form.

$$y'' = f(y, y', x) \quad \ldots\ldots (2.\ 1.\ 1.)$$

with the two point boundary conditions

$$y(0) = a_1 \qquad y(b) = a_2$$

We posses no convenient or useful technique for representing general solution in terms of a finite set of particular solution as in the linear case. Consequently we posses no ready means of reducing the transcendental problem in soling (2. 1. 1) to an algebraic problem as is the situation in case f(y, y', x) is linear in y and y'.

To obtain an analytic foot hold and simultaneously to provide computational algorithms, we must have recourse to approximation techniques. Fixed point methods so valuable in establishing existence of solutions are of no use numerically. Generally few of the standard classical techniques as successive approximations are of much utility numerically. None the less of now a number of powerful computational methods exists. We shall now study the quasilinearization technique.

Consider the second order nonlinear equation

$$y'' = f(y',y,x) \quad \ldots\ldots (2.\ 1.\ 2)$$

With nonlinear boundary conditions of the form

$$g_1 [y(0), y'(0)] = 0 \qquad g_2 [y(b), y'(b)] = 0$$

Or even more generally

$$\left.\begin{aligned} g_1[y(0),y'(0),y(b),y'(b)] = 0 \\ g_2[y(0),y'(0),y(b),y'(b)] = 0 \end{aligned}\right\} \quad \text{. . . . (2. 1. 3)}$$

We can now apply quasilinearization to both equation and the boundary conditions. Thus in the case of equation (2. 1. 2) subject to the conditions (2. 1.3) we generate sequence (y_n (x)) by means of the equation.

$$Y''_{n+1} = f_{y'}(y'_n , y_n, x) (y'_{n+1} – y'_n) + f_y (y'_n, y_n, x) (y_{n+1} – y_n) + f(y'_n, y_n, x) \quad \text{. . . . . (2. 1. 4)}$$

With the liberalized

Boundary conditions

$g_{1y}[y_n(0), y'_n(0)] [y_{n+1} (0) – y_n (0)] + g_{1y'} [y_n(0), y'_n(0)] [y'_n+1 (0) – y'_n(0)]=0$ . . . . . (2.1.5.)

and a similar equation can be derived from g_2 for the point x=b

Consider the example

$$Y'' = - y + \frac{2(y')^2}{y}$$

$$Y(-1) = y(1) = 0.324027 \quad \text{. . . . . (2. 1. 6)}$$

Here $f(x, y, y') = -y + \dfrac{2(y')^2}{y}$

Consider the Taylor series expansion of (2. 1. 6)

$$y''_{n+1} = f(x_n, y_n, y'_n) + (y'_{n+1} - y'_n) fy' \quad \text{. . . . . (2. 1. 7)}$$

We have $fy = -1 -2 (y')^2 / y^2$

And $f_{y'} = 4y' / y$

Put n = 0, substituting these values in (2. 1. 7)

$$y''_1 = -y_0 + \frac{2(y'_0)^2}{y_0} + (y_0 - y_0)\left\{-1 - \frac{2(y'_0)^2}{{y_0}^2}\right\} + (y'_1 - y'_0)4\frac{y'_0}{y_0} \quad \text{. . . . . (2. 1. 8)}$$

We have boundary conditions y(-1) = y(1) = 0.324027

$$Let.y_0(x) = Ax + B$$

$$y_0^{(-1)} = -A + B = 0.324027$$

$$y_0(1) = A + B = 0.324027$$

Solving these two equations we get

A=0, B=0.324027

$$\therefore y_0(x) = 0.324027$$

$$y'_0(x) = 0$$

Substituting these values in equation (2. 1. 8) we get

$y''_1 = -0.324027 + [y_1 – 0.324027] (-1)$

$y''_1 = - 0.324027+0.324027 – y_1$

i.e. in the form $y'' = - y$ . . . . (2. 1. 9)

is the required linear equation with the boundary conditions y(-1) = y(1) = 0.324027

The numerical solution of this equation is computed by three methods namely shooting method, finite difference method and finite element method and the details are given in chapter 3 section 2, as example 2.

2.2 SHOOTING METHOD.

One of the very popular approaches to solve a two point boundary value problem is to reduce it to a problem in which a program for solving initial value problem can be used.

In the shooting method we create an initial value problem by assuming a sufficient number of initial values. Solve this initial value problem and compare the computed value with the given conditions at the other boundary. Repeat the solution with varying values of assumed conditions until agreement is attained at the other boundary.

Consider the two point boundary value problem

Y''=f(x, y, y') a<x<b

$$A\begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} + B\begin{bmatrix} Y(b) \\ Y'(b) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \quad \text{.....(2.2.1)}$$

The terms A and B denote given square matrices of orders 2x2 and γ_1 and γ_2 Are given constants.

The theory for the nonlinear problems is far complicated than that for the linear problems. We given an introduction to the theory for the following more limited problems.

$$\gamma'' = f(x, y, y')$$
$$a_0\, y(a) - a_1 y'(a) = \gamma_1$$
$$b_0 y(b) + b_1 y'(b) = \gamma_2$$

We now develop a method for the boundary value problem (2.2.2) consider the initial value problem

$$y'' = f(x, y, y')$$
$$y(a) = a_1 s - c_1\gamma_1$$
$$y'(a) = a_0 s - c_0\gamma_1$$

Depending on the parameter s, where c_0 and c_1 are arbitrary constants satisfying

$$a_1 c_0 - a_0 c_1 = 1$$

Denote the solution of (2.2.3) by y(x,s) then it is straight forward to see that

$$a_0 y(a,s) - a_1 y'(a,s) = \gamma_1$$

For all s for which y exists.

Since y is a solution of (2.2.1) all that is needed for it to be a solution of (2.2.1) is to have it satisfy the remaining boundary condition at b.

This means that y(x,s) must satisfy

$$\phi(s) \equiv b_0 y(b,s) + b_1 y'(b,s) - \gamma_2 = 0 \quad \text{........(2.2.4)}$$

This is a nonlinear equation for s. if s^* is a root of $\phi(s)$. Then y(x,s) will satisfy the boundary value problem (2.2.1). it can be show that under suitable assumption of f and it's boundary conditions (2.2.3) will have unique solution s^*. We can use a roof finding method for nonlinear equations to solve for s^*.

The method is called shooting method because is resembles artillery problems artillery problem. One sets the elevation of the gun fires a preliminary round at the target one zero's in on it by using intermediate of the guns elevation.

Any of the root finding methods can be applied to solve $\phi(s) = 0.$ Each evaluation of $\phi(s)$ involves the solution of the initial value problem (2. 2. 3) over [a, b] and consequently we want to minimize the number of such evaluations. As a specific example of an important and rapidly convergent method we look at new tons method.

$$S_{m+1} = S_m + \frac{\phi(s_m)}{\phi'(s_m)} \quad m=0, 1, \ldots\ldots \qquad \ldots.(2.2.5)$$

To calculate $\phi'(s)$ differentiate (2. 2. 3) to obtain

$$\phi'(s) = b_o\zeta(b) + b_1\zeta_s'(b) \qquad \ldots\ldots(2.2.6)$$

Where $\zeta_s(x) = \frac{\partial y(x,s)}{\partial s}$ ……(2. 2. 7)

To find $\zeta_s(x)$ differentiate the equation

Y''' (x, s) = f[x, y(x,s), y'(x,s)]

With respect to s.

Then ζ_s satisfies the initial value problem

$$\zeta_s''(x) = f_2[x, y(x,s), y'(x,s)]\zeta_s'(x) + f_3[x, y(x,s), y'(x,s)]\zeta_s'(x) \quad \ldots(2.2.8)$$

$$\zeta_s(a) = a_1 \qquad \zeta_s'(a) = a_0$$

The functions f_2 and f_3 denote partial derivatives of f(x, u, v) with respect to u and v. The initial values are those obtained in (2. 2. 3) and from the definition of ζ_s.

The procedure of shooting method is developed as an algorithm in chapter 3 section 1 and used for solving examples.

2.3 FINITE DIFFERENCE METHOD

There exists many methods for solving second order boundary value problem. Of these finite difference is a popular one.

Consider a two point boundary value problem

Y''(x) + f(x) y'(x) + g(x) y(x) = r(x) ….(2. 3. 1)

With the boundary conditions

$Y(x_0) = a$ $\qquad$ $y(x_n) = b$ ….(2. 3. 1)

The finite difference method for the solution of a two point boundary value pr oblem consists of replacing the derivatives occurring in the differential equation (and the boundary conditions as well) by means of their finite difference approximations and then solving the resulting system of equations by a standard procedure.

To obtain appropriate finite difference approximations to the derivatives expand y9x+h) in Taylors series we have y(x+h) = y(x) + h y'(x) + $\frac{h^2}{2}y''(x) + \ldots\ldots$ …(2. 3. 3)

From which we obtain

$$y'(x) = \frac{y(x+h) - y(x)}{h^2} - \frac{h}{2}y''(x)$$

Thus we have $y'(x) = \frac{y(x+h) - y(x)}{h^2} + 0(h)$ ..(2. 3. 4)

Similarly expanding y(x-h) in Taylors series givens

$$y(x-h) = y(x)y'(x) + \frac{h^2}{2}y''(x) + \qquad \ldots\ldots(2.3.5)$$

From which we obtain

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$$y(x) = \frac{y(x) - Y(x - h)}{h} + 0(h) \qquad \text{. . . . (2. 3. 6)}$$

A Central difference approximation for y'(x) can be obtained by subtracting (2. 3. 5) from (2. 3. 3) we thus have

$$y'(x) = \frac{y(x+h) - (x-h)}{2h} + 0(h^2)$$

It is clear that (2.3.7) is a better approximation to $y^1(x)$ than earlier. Again adding (2.3.3) and (2.3.5) we get an approximation for y''(x)

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{2h} + 0(h^2) \qquad \text{……(2.3.3)}$$

In a similar manner it is possible to derive finite difference approximations to higher derivation. To solve the boundary value problem definer by (2.3.1) and (2.3.2) we divide the rang $[x_0,x_n]$ into n equal sub intervals of width h so that

$X_i = x_0 + ih$ $\qquad$ I = 1,2,3……..n

The corresponding value of y at these points are denoted by

$Y(x_i) = y_i = y(x_0+ih)$ $\qquad$ i=1,2……..,n

From equations (2.3.7) and (2.3.8) value of y' (x) and y'' (x) at the point $x=x_1$ can now be written as

$$y'_1 = \frac{y'_{i+1} - y_{i-1}}{2h} + 0(h^2)$$

And $$y''_{i1} = \frac{y_{i1+1} - 2y_i + y_{i-1}}{h^2} + 0(h^2)$$

Satisfying the differential equation at the point $x = x_1$ we get

$$y''_i + f_i y'_i + g_i y_i = r_i$$

Substituting the expression for y'_1 and y''_1 this gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + f_i \frac{y_{i+1} - y_{i-1}}{2h} + g_i y_i = i_1$$

i=1, 2, ... n-1

Where $\qquad y_i = y(x_i)$

$g_i = g(x_i)$ $\qquad$ etc.

Multiplying through out by h_2 and simplifying we get

$$(1 - \frac{h}{2} fi) y_{i-1} + (-2 + gih^2) y_{i1} + (1 + \frac{h}{2} fi) y_{i+1} = rih^2 \qquad \text{. . . . (2. 3. 9)}$$

i = 1, 2, ... n-1

With $\qquad y_0 = a$

$y_n = b$ $\qquad$ . . . . . (2. 3. 10)

Equation (2. 3. 9) with (2. 3. 10) comprises a tri diagonal system which can be solved.

The solution of tridiagonal system constitute an approximate solution of the boundary value problem defined by (2. 3. 1) and (2. 3. 2.)

The algorithm for above procedure is presented in chapter 3 (3. 1c).

2.4. FIINITE ELEMENT METHOD

In the finite difference approximation of a differential equation. The derivatives in the equation are replaced by difference quotients which involves the values of the solution at discrete mesh points of the domain. The resulting discrete equations are solved after imposing boundary conditions for the values of the solution at the mesh points. Although the finite difference method is simple concept its suffers from several disadvantages. The most notable are the inaccuracy of the derivatives of the approximated solution, the difficulty in imposing the boundary conditions along non-straight boundaries, the difficulty in accurately representing geometrically complex domains and the inability to employ no uniform and non rectangular meshes.

The finite element method over comes the difficulty of the variational methods because it provides a systematic procedure for the derivation of the approximation functions.

The finite element method is an approximate method of solving differential equations of boundary and/or initial value problems in engineering and mathematical physics. In this method a continuum is engineering and mathematical physics. In this method a continuum is divided into many small elements of convenient shapes choosing suitable points called nodes with in the elements. The variable in the differential equation is written as a linear combination of appropriately selected interpolation functions and the values of the ariable or its various derivatives specified at the nodes. Using ariational principles or weighted residual methods the governing differential equations are transformed into finite element equations governing all isolated elements. These local elements are finally collected together to form a global system of differential or algebraic equations with proper boundary and/or initial conditions imposed and hence solved.

GALERKIN METHOD TO DERIVE FINITE ELEMENT EQUATIONS

Consider the differential equation

$$\frac{d^2y}{dx^2} - \alpha^2 y - f(x) = 0$$

By substituting the approximarte function into this differential equation we expect to have commited an error or a residual ε. Thus we any write.

$$\frac{d^2y}{dx^2} - \alpha^2 y - f(x) = \varepsilon \qquad \text{. . . . (2. 4. 1.)}$$

We construct an inner product of this residual and the global finite element interpolation function ϕ_1.

$$(\varepsilon, \phi_1) = {}_0\int^1 \left[\frac{d^2y}{dx^2} - \alpha^2 y - f(x) \right] \phi_1 dx = 0 \qquad \text{. . . . . (2. 4. 2.)}$$

This an orthogonal projection of the residual space on to a subspace spanned by ϕ_1. Integrating (2. 4. 2) by parts yields.

$$\frac{dy}{dx}\phi_1 \Big|_0^1 - {}_0\int^1 \left[\frac{dy}{dx}\frac{d\phi_1}{dx} + \alpha^2 y \phi_1 + f(x)\phi_1 \right] dx = 0 \qquad \text{. . . . . (2. 4. 3)}$$

The boundary term obtained here is the natural boundary condition. We note that the interpolation function ϕ_1 does not include the boundary. If a two dimensional problem were considered, we would have required two types of interpolation functions : one for the interior domain and other for the boundary surfaces; that is

$$Y(x, y) = \phi_1 (x, y)\, y_1 \qquad \text{. . . . .(2. 4. 4)}$$

And

$$Y(\Gamma) = \phi_k^* (\Gamma)\, y_k \qquad \text{. . . . (2. 4. 5)}$$

Where i denotes all interior global nodes in Ω and k denotes all boundary conditions along Γ. Clearly ϕ_k^* (r) is the interpolation function which represents the variation of dy/dn along the boundary surface (line) so that the global boundary integral of the type.

$$\int_\Gamma \frac{dY}{dn}\phi_k^*(\Gamma)d\Gamma = \sum_{e=1}^{E}\int_0^a \frac{dy^{(e)}}{dn}\phi_n^{*(e)}\Delta_{NK}^{(e)} d\Gamma$$

(N = boundary element nodes) . . . . . . (2. 4. 6)

Can be performed as the unior of each of the boundary elements. However in a one dimensional problem there exists no boundary surface; there are two boundary points, one at each end of the

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domain. Returning to the boundary term (2. 4. 3) if dy/dx is specified at ends, ϕ_1 must be the boundary interpolation ϕ_k^* is simply a unity.

$$\phi_1^* = \Delta_{N1}^{(e)}\phi_N^{*(e)} = 1, \qquad \phi_1^*(Z_j) = \delta_{1j},$$

$$\phi_1^{*(e)}(Z_M) = \delta_{NM} \qquad \text{. . . . . (2. 4. 7)}$$

Here i, j and N, M represents the boundary nodes for the global and local system with onlyu boundary element and boundary node being involved. Therefore, rewrite (2. 4. 3) in the form

$$\int_0^1 \left[\frac{dY}{dx}\frac{d\phi_1}{dx} + \alpha^2 y\phi_1 + f(x)\phi_1 \right] dx = \frac{dY}{dx}\phi_1 \Big|_0^1$$

And

$$A_{1j}Y_j = F_1 + \frac{dY}{dx}\phi_1^* \Big|_{x=0,x=1} = F_1 + \frac{dY}{dx}\Big|_{1(x=0,x=1)}$$

If the given problem is the Dirichlet type, then we simply have A_{1j} $Y_j = F_1$. . . . . (2. 4. 8)

Where

$$A_{1j} = \int_0^1 \left[\frac{d\phi_1}{dx}\frac{d\phi_j}{dx} + \alpha^2\phi_1\phi_j \right] dx \qquad \text{. . . . (2. 4. 9)}$$

And

$$F_1 = -\int_0^1 f\phi_1 dx \qquad \text{. . . . (2. 4. 10)}$$

Here A1j is nxn positive definite matrix. The equation (2. 4. 8) is called the global finite element equations. It may be said that the global eqations (2. 4. 8) represent the collection or assembly of local equations, A glance at (2. 4. 9) and (2. 4. 10) indicates that the local element matrices $A^{(e)}_{NM}$ and the local inpur vector $F^{(e)}_N$ are assembled according to the Boolean matrices which place the appropriate local nodal contributions to the corresponding global system. Equation (2. 4. 8) comprise a tridia system which can be solved by the method outlined in chapter-1 section 3. The solution of this traditional system constitutes an approximate solution of the boundary value problem. The algorithm for this method is presented in chapter 3 (3. 1d).

CHAPTER – 3

3.1 ALGORITHMS

In this chapter the algorithms of 1) Solution of tri diagonal method 2) Shooting method 3) Finite difference method 4) Finite element method have been given. These algorithms are used in writing computer programs which are employed in obtaining numerical solutions of examples :

Example 1 :

With y’’ = y

Y(0) = 1

Y(2) = 7.38905

Example 2 :

$$y = - y + \frac{2(y')^2}{y}$$

With y(-1) = 0.324027

y(1) = 0.324027

Example 3 :

$$y = x + (1 - \frac{x}{5})y$$

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With y(1) = 2

y(3) = -1

3.13 SOLVING A TRIDIAGONAL SYSTEM OF EQUAITONS

A Computational procedure due to Thomas to solve tridiagonal systems of equations represented by matrix (1. 3. 2) is given below.

* Solving tridiagonal systems of equations

(i) Set $\alpha_1 = b_1$ and compute

$$\alpha_1 = b_1 - \frac{a_1 c_{1-1}}{\alpha_{1-1}} \qquad 1 = 2, 3, \ldots . n$$

(ii) Set $\beta_1 = \frac{d_1}{b_1}$ and compute

$$\beta_1 = \frac{d_1 - a_1\beta_{1-1}}{\alpha_1} \qquad 1 = 2, 3, \ldots . . n$$

(iii) Finally, Compute u_1 from

$$u_1 = \beta_1 - \frac{c_1 u_{1+1}}{\alpha_1} \qquad 1 = n\text{-}1, n\text{-}2, \ldots . 1$$

Where $u_1 = \beta_1$

This p ro c e dure has been found to be ery efficient for use on a digital computer.

3.1b SHOOTING METHOD

* To solve a second order BVP by Shooting method

* The problem is y'' = F(x, y, y')

* With $y(a_1) = b_1$ $y(a_2) = b_2$

Read h, g_1, g_2, a_1, b_1

xf = a_2

$x_0 = a_1$

$y_0 = b_1$

$dy_0 = g_1$

Call rkm (F, k_1, k_2, y, dy, x)

Print x, y

If $(x \geq xf)$ GOTO 10

10 r_1 = y

Now $dY_0 = g_2$

Call rkm (F, k_1, k_2, y, dy, x)

Print x, y

If $(x \geq xf)$ GOTO 20

20 r_2 = y

D = b_2

$dy_0 = g_1 + (g_2 - g_1)(D-r_1)/(r_2-r_1)$

Call rkm (F, k_1, k_2, y, dy, x)

Print x, y

If $(x \geq xf)$ GOTO 30

30 Stop

End

Subroutine rkm (F, k_1, k_2, y, dy, x)

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40 $k_1 = \frac{h^2}{2_2} F(x_0, y_0)$

$$k_1 = \frac{h^2}{2_2} F(x_0 + \frac{2}{3}h, y_0 + hdy_0 \frac{2}{3} + \frac{4}{q} k_1)$$

$y_1 = y_0 + h\ dy_0 + (k_1 + k_2)/2$
$dy_1 = dy_0 + (k_1 + 3k_2)/2h$
$x = x_0 + h$
Print x, y_1
Now $x_0 = x$
$y_0 = y_1$
$dy_0 = dy_1$
Return
End
Function F(x, y, y')
Return
End

3.1c FINITE DEFERENCE METHOD

C To Solve a second order BVP by finite difference method
The given equation is y'' + fy' + gy = r
With $y(a_1) = b_1$ and $y(a_2) = b_2$
$h = (a_2 – a_1) / n$
1 = n-1
Do for i = 1, n

$$a(1) = (1 - \frac{h}{2} f)$$

$$b(1) = (gh^2 - 2)$$

$$c(1) = (1 + \frac{h}{2} f)$$

$$d(1) = rh^2$$

End to loop

Set $\alpha(1) = b(1)$ and co mpute

$$\alpha(1) = b(1) - \frac{a(1)c(1-1)}{\alpha(1-1)}$$ for 1=2, 3, . . . n

Set $\beta(1) = d(1)/b(1)$ and compute

$$\beta(1) = d(1) - \frac{a(1)\beta(i-1)}{\alpha(i)}$$ for i = 2, 3, ... n

Finally compute

$$u(1) = \beta(1) - \frac{c(1)u(1+1)}{\alpha(1)}$$ for 1 = n-1, n-2, . . . 1

Where u(n) = $\beta(n)$
Stop
End

3.1d FINITE ELEMENT METHOD

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C To solve a second BVP by Finite element method
The given problem is y'' = F(x, y, y')
Read n
do for 1=1, 2
do for j=1, 2
$a(1,1)= f_1(\alpha,h)$
$a(2,2)= f_2(\alpha,h)$
$a(1,2)= g_1(\alpha,h)$
$a(2,1)= a(1,2)$
End do loop
do for 1=2, 3, ... n-1
do for j=2, 3, ...n-1
b(1, 1) = a(1, 1)+a(2, 2)
b(1, j+1)=a(2,1)
b(1+1, j) = a(2, 1)
End of do loop
do for 1 = 1, 2, ... n
b(1) = b(1, 1)
a(1) = b(1+1, j)
c(1) = b(1, j+1)
d(1) = f(1)

c. Solution by solving this tridiagonal system
Set $\alpha(1)=b(1)$ and compute

$$\alpha(1)=b(1)-\frac{a(1)c(1-1)}{\alpha(1-1)}$$ for 1=2, ... n

Set $\beta(1)=d(1)/b(1)$ and compute

$$\beta(1)=d(1)-\frac{a(1)\beta(1-1)}{\alpha(1)}$$ for 1 = 2, 3, ...n

Finally compute

$$u(1)=\beta(1)-\frac{c(1)u(1+1)}{\alpha(1)}$$ for 1 = n-1, n-2 1

Where u(n) = $\beta(n)$
Stop
End

3.2 NUMERICAL SOLUTIONS OF EXAMPLES

In this section the examples 1, 2 and 3 with actual solutions (where ever possible) and numerical solutions using shooting method, finite difference method and finite element method in forms of tables are presented.

Example 1:

Y'' = y

With y(0) = 1
Y(2) = 7.38905

Analytical solution is $y = e^x$

Example 2 :

$$y'' = -y + \frac{2(y')^2}{y}$$

With y(-1) = 0.324027

Y(1) = 0.324027

Qusilinearizatioin technique (vide 2.1 equation no (2. 1. 9) reduced this equation to the form

Y’’ = -y   y(-1) = y(1) = 0.324027

Analytical solution is $\frac{1}{e^x + e^{-x}}$

Example 3:

$$Y'' = x + (1- \frac{x}{5})y$$

Y(1) = 2

Y (3) = -1

Solution as given by shooting method has been taken for computing errors in the solutions by FDM and FEM.

SOLUTION of y’’=y   y(0) = 1   y(2) = 7.389
BY SHOOTING METHOD

X	Y	Exact SOL	ERROR
0.2000	**1.22146**	**1.22140**	**0.00006**
0.40000	**1.22146**	**1.49182**	**0.00010**
0.60000	**1.82226**	**1.82212**	**0.00014**
0.80000	**2.22572**	**2.22554**	**0.00017**
1.00000	**2.71848**	**2.71828**	**0.00019**
1.20000	**3.32032**	**3.32012**	**0.00020**
1.40000	**4.05538**	**4.05520**	**0.00018**
1.60000	**4.95318**	**4.95303**	**0.00014**
1.80000	**6.04972**	**6.04965**	**0.00007**
2.00000	**7.38900**	**7.38906**	**0.00006**

Y’’ = y   y(0)=1 y(2)=7.3890
FINITE DIFFERENCE METHOD

X	ACT. SOL	Y	ERROR
0.00	1.00000	1.00000	0.00000
0.20	1.22140	1.22236	0.00096
0.40	1.49182	1.49361	0.00179
0.60	1.82212	1.82461	0.00249
0.80	2.22554	2.22859	0.00305
1.00	2.71828	2.72171	0.00343
1.20	3.32012	3.32371	0.00359
1.40	4.05520	4.05865	0.00345
1.00	4.95303	4.95594	0.00290
1.80	6.04965	6.05146	0.00182

To Solve y’’=y by FEM
Y(0)=1
Y(2)=7.38905.

Y	X
0.89960	0.2
0.83542	0.4
1.29502	0.6
1.80676	0.8
2.39125	1.0
3.07202	1.2
3.87649	1.4
4.83705	1.6
5.99238	1.8

Solution of $y'' = -y + \frac{2y'^2}{y}$ $y(-1) = y(1) = 0.324027$

BY SHOOTING METHOD

X	Y	EXACT SOL	ERROR
-0.80000	0.41783	0.37385	0.04398
-0.60000	0.49498	0.42178	0.07320
-0.40000	0.52539	0.46250	0.08988
-0.20000	0.58777	0.49016	0.09761
0.00000	0.59972	0.50000	0.09972
0.20000	0.58776	0.49016	0.09760
0.40000	0.55237	0.46250	0.08987
0.60000	0.49496	0.42178	0.07319
0.80000	0.41782	0.37385	0.4397
1.00000	0.32403	0.32403	0.00000

To sole the equation by FEM

$Y'' = -y + \frac{2y'^2}{y}$ $y(-1) = y(1) = 0.324027$

X	Y
-0.8	0.3517
-0.6	0.3868
-0.4	0.4189
-0.2	0.4417
0.0	0.4467
0.2	0.4401
0.4	0.4195
0.6	0.3809
0.8	0.3407

$y''=y+2y'^2/y$ $y(-1)=y(1)=0.324027$

Finite Difference Method

X	ACT. SOL	Y	ERROR
-1.00	0.32403	0.32403	0.00000
-8.00	0.37385	0.41834	0.04449
-0.60	0.42178	0.49592	0.07414
-0.40	0.46250	0.55366	0.09116

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-0.20	0.49016	0.58926	0.09909
0.00	0.50000	0.60128	0.10128
0.20	0.49016	0.58926	0.09909
0.40	0.46250	0.55366	0.09116
0.60	0.42178	0.49592	0.07414
0.80	0.37385	0.41834	0.04449

SOLUTION OF y" = x+(1-x/5)y, y(1)=2, y(3)+-1
BY SHOOTING METHOD

X	y
1.20000	1.35029
1.40000	0.79003
1.60000	0.30886
1.80000	-0.09964
2.00000	-0.43843
2.20000	-0.70753
2.40000	-0.90429
2.60000	-1.02369
2.80000	-1.05859
3.00000	-1.00000

Y" = x+1(1-x/5)y y(1) =2 y(3) = -1
FINITE DIFFERENCE METHOD

X	Y
1.00	2.00000
1.20	1.35133
1.40	0.79175
1.60	0.31097
1.80	-0.09736
2.00	-0.43617
2.20	-0.70546
2.40	-0.90254
2.60	-1.02240
2.80	-1.05789

TO SOLVE THE EQUATION Y = X+ (1-X/5)Y
Y(1) = 2, Y(3)= - 1 BY FEM

X	Y
1.2	1.2785
1.4	0.7203
1.6	0.2175
1.8	-0.2691
2.0	-0.3439
2.2	-0.6359
2.4	-0.8347
2.6	-0.9613
2.8	-1.008

TABLES AND OBSERVATION

The following are the consolidated table showing the solutions of each example by three methods and errors are Y(computed)- Y(analytical solution).

Example-1: y” =y given y(0)=1 and y(2)=7.38905

X	Analytical solution $Y=e^x$	Error by Shooting method	Error by FDM	Error by FEM
0.0	1.00000	0.00000	0.00000	0.0000
0.2	1.22140	0.00006	0.00096	0.32172
0.4	1.49182	0.00010	0.00179	0.65420
0.6	1.82212	0.00014	0.00249	0.52711
0.8	2022554	0.00017	0.00305	0.41878
1.0	2.71828	0.00019	0.00343	0.32703
1.2	3.32012	0.00020	0.00359	0.24812
1.4	4.05520	0.00018	0.00345	0.17871
1.6	4.95303	0.00014	0.00290	0.1198
1.8	6.04965	0.00007	0.00290	0.05627
2.0	7.38905	0.00006	0.00000	0.00000

Example-2: $y'' = -y + (2y'^2/y)$ given y(-1)= y(1)=0324027

X	Analytical solution $Y=(e^x + e^{-x})^{-1}$	Error by Shooting method	Error by FDM	Error by FEM
-1.0	0.32403	0.00000	0.00000	0.0000
-0.8	0.37385	0.04398	0.04449	0.02215
-0.6	0.42178	0.07320	0.07414	0.03498
-0.4	0.46250	0.08988	0.09116	0.04450
-0.2	0.49016	0.09761	0.09909	0.04846
0.0	0.50000	0.09972	0.10128	0.05332
0.2	0.49016	0.09760	0.09909	0.05006
0.4	0.46250	0.08987	0.09116	0.04327
0.6	0.42178	0.07319	0.07414	0.04088
0.8	0.37385	0.04397	0.04449	0.03315
1.0	0.3240.	0.00000	0.00000	0.00000

Example-2: y” = x+ (1-x/5)y given y(1)= 2 & y(3)=-1

x	Solution by SHM	Solution by FDM	Solution by FEM
1.0	2.00000	2.00000	2.0000
1.2	1.35029	1.35133	1.2785
1.4	0.79003	0.79175	0.7203
1.6	0.30886	0.31097	0.2175
1.8	-0.09984	-0.09736	-0.2691
2.0	-0.43843	-0.43617	-0.3439
2.2	-0.70753	-0.70546	-0.6539
2.4	-0.90429	-0.90254	-0.8347
2.6	-1.02369	-1.02240	-0.9613
2.8	-1.05859	-1.05789	-1.0080
3.0	-1.00000	-1.0000	-1.0000

The shooting method is often quite laborious. Especially with problem of fourth order and higher order equations, the necessity to assume two or more conditions at the starting point is slow and tedious. It involves some sort of risk of wasting time on making assumptions. The finite difference method can be considered as direct discretization of differential equations. In finite element methods difference equations using approximate methods have been generated with piece wise polynomial solution. The numerical solution by these methods in case of each example have been presented.

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